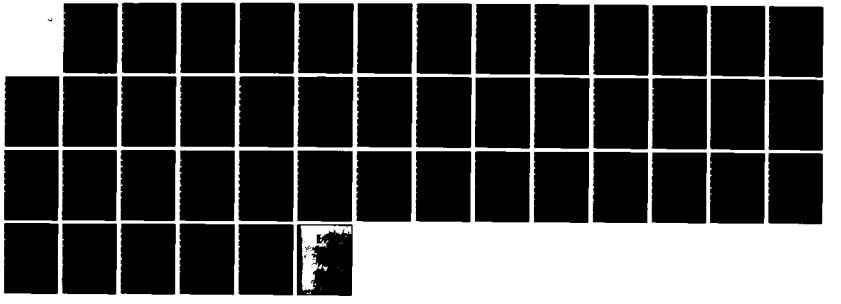
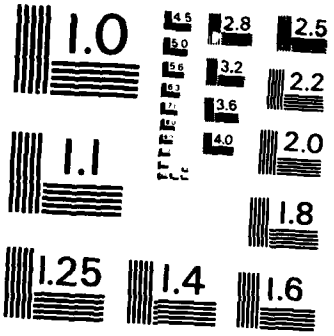
AD-A142 328 AN INVERSE LAMINAR BOUNDARY LAYER PROBLEM WITH ASSIGNED 1/1
WALL SHEAR: THE M.. (U) PENNSYLVANIA STATE UNIV
UNIVERSITY PARK APPLIED RESEARCH LAB..
UNCLASSIFIED K C KAUFMAN ET AL. 04 MAY 84 F/G 20/4 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

AD-A142 328

AN INVERSE LAMINAR BOUNDARY LAYER PROBLEM WITH ASSIGNED WALL SHEAR: THE MECHUL FUNCTION REVISITED

K. C. Kaufman and G. H. Hoffman

Technical Memorandum
File No. TM 84-79
4 May 1984
Contract N00024-79-C-6043

Copy No. 10

The Pennsylvania State University
Intercollege Research Programs and Facilities
APPLIED RESEARCH LABORATORY
Post Office Box 30
State College, Pa. 16804

NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND

DTIC FILE COPY

This document has been approved for public release and sale; its distribution is unlimited.

84 06 22 063

AN INVERSE LAMINAR BOUNDARY LAYER PROBLEM WITH ASSIGNED WALL SHEAR: THE MECHUL FUNCTION REVISITED

K. C. Kaufman and G. H. Hoffman

ARL/PSU

Technical Memorandum
File No. TM-84-79
4 May 1984
Contract N00024-79-C-6043

Copy No. 10

The Pennsylvania State University
Intercollege Research Programs and Facilities
APPLIED RESEARCH LABORATORY
Post Office Box 30
State College, PA 16804

Approved for Public Release
Distribution Unlimited

NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND

This document has been approved for public release and sale; its distribution is unlimited.

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1 REPORT NUMBER TM 84-79	2. GOVT ACCESSION NO. AD-A142	3 RECIPIENT'S CATALOG NUMBER 325
4. TITLE (and Subtitle) AN INVERSE LAMINAR BOUNDARY LAYER PROBLEM WITH ASSIGNED WALL SHEAR: THE MECHUL FUNCTION REVISITED		5 TYPE OF REPORT & PERIOD COVERED Technical Memorandum 6 PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) K. C. Kaufman and G. H. Hoffman		8. CONTRACT OR GRANT NUMBER(s) N00024-79-C-6043
9. PERFORMING ORGANIZATION NAME AND ADDRESS Applied Research Laboratory Post Office Box 30 State College, PA 16804		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Sea Systems Command, Code NSEA 63R31 Washington, DC 20362		12. REPORT DATE 4 May 1984 13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) UNCLASSIFIED 15a. DECLASSIFICATION DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release. Distribution unlimited.
Per NAVSEA - 13 June 1984

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

inverse boundary layer
numerical solutions
viscous flow

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

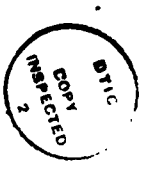
An inverse method is presented for accurately determining the pressure gradient distribution for specified wall shear in a two-dimensional laminar boundary layer. In the inverse formulation, the pressure gradient is treated as a dependent variable that is only a function of the streamwise coordinate. The method presented here is a reformulation of the mechul function scheme of Cebeci and Keller with additional techniques to increase the stability of the solution. These techniques

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

include the use of fourth order splines to approximate normal derivatives and three point backward finite differences for the streamwise derivatives. Partial pivoting is also used in the solution of the block tridiagonal system resulting from the linearized equations of motion. The solution is obtained using the Newton iteration method. Numerical examples are presented for self-similar and non-similar solutions.



From: K. C. Kaufman and G. H. Hoffman

Subject: An Inverse Laminar Boundary Layer Problem with Assigned Wall Shear: The Mechul Function Revisited

References: See p. 25

Abstract: An inverse method is presented for accurately determining the pressure gradient distribution for specified wall shear in a two-dimensional laminar boundary layer. In the inverse formulation, the pressure gradient is treated as a dependent variable that is only a function of the streamwise coordinate. The method presented here is a reformulation of the mechul function scheme of Cebeci and Keller with additional techniques to increase the stability of the solution. These techniques include the use of fourth order splines to approximate normal derivatives and three point backward finite differences for the streamwise derivatives. Partial pivoting is also used in the solution of the block tridiagonal system resulting from the linearized equations of motion. The solution is obtained using the Newton iteration method. Numerical examples are presented for self-similar and non-similar solutions.

Table of Contents

	Page
Abstract	1
List of Figures	3
List of Tables	4
Nomenclature	5
I. Introduction	6
II. Analysis	7
Governing Differential Equations	7
Differential Equation in First Order Form	10
Streamwise Discretization	11
Normal Discretization	12
Linearization	15
Two Point Spline Relations at the Wall	17
Two Point Spline Relations at the Outer Edge	18
Solution of Block Tridiagonal Matrix Equations	19
Starting Solution	19
III. Results and Discussion	21
Similar Flow Problem	21
Non-Similar Flow Problem	22
A Note on Normal Stepsize	23
IV. Conclusion	24
References	25
Tables	26
Figures	32
Appendix A	36

List of Figures

		Page
Figure 1.	Schematic of Computational Domain	32
Figure 2.	Boundary Layer Velocity Distribution for Self-Similar Flows	33
Figure 3.	Pressure Gradient Parameter as a Function of ξ: Case 1 . . .	34
Figure 4.	Pressure Gradient Parameter as a Function of ξ: Case 2 . . .	35

List of Tables

		Page
Table I.	Comparison of Positive Wall Shear Solutions for Self-Similar Flows	26
Table II.	Comparison of Reverse-Flow Solutions for Self-Similar Flows	27
Table III.	Computed Pressure Gradient Parameter β as a Function of ξ for Case 1	28
Table IV.	Comparison of β as a Function of ξ with Varying η_∞ for Case 1	29
Table V.	Computed Pressure Gradient Parameter β as a Function of ξ for Case 2	30
Table VI.	Comparison of β as a Function of ξ with Varying η_∞ for Case 2	31

Nomenclature

η	normal boundary layer coordinate
ξ	streamwise boundary layer coordinate
f	dimensionless stream function variable
S	specified wall shear
N	number of steps in the normal direction
M	number of steps in the streamwise direction
L_j, P_j, Q_j, T_j	right hand side groups for spline correction equations containing known quantities from previous iteration or streamwise stations
A_j, B_j, C_j	4×4 block tridiagonal coefficient matrices
R_j	column vector of known quantities
Z_j	solution correction vector
ε	convergence criteria value
n	iteration number
i	streamwise grid location
j	normal grid location

All other quantities are defined in the text.

I. Introduction

The object of the present study is the accurate numerical determination of the pressure gradient distribution for specified wall shear in a laminar boundary layer. This is known as the inverse problem. This procedure is sometimes preferable to the standard problem, in which the pressure distribution is prescribed and the body shear stress is determined by the solution of the boundary layer equations.

One method of determining the pressure gradient distribution is the mechul function scheme, developed by Cebeci and Keller [1]. This scheme was found to be accurate and efficient for self-similar type flow problems. For non-similar flow problems, however, this method exhibits a weak numerical instability. The instability appears as a disturbance that develops in the far field of the computational domain and propagates towards the wall. As a result, the numerical solution is ultimately destroyed as it is marched downstream.

The inverse problem requires the determination of a coefficient function, the pressure gradient parameter, which satisfies an over-determined set of boundary conditions that result when the wall shear is specified. The mechul function scheme adds an extra differential equation for the pressure gradient to the system of boundary layer equations. As a result, the system is no longer over-determined, and the governing equations may be solved as an extention of the standard problem.

This study reformulates the mechul function scheme as developed in Ref. [1] in such a way that the numerical instability is eliminated, thus yielding an accurate solution method. Whereas in Ref. [1], the Keller box scheme was used to discretize the differential equations, the new procedure applies fourth order splines developed by Rubin and Khosla [3] to approximate the normal derivatives and three point backward differences for the streamwise derivatives. This yields a fully implicit method which lends stability to the solution.

In the present work, the discretized, linearized system of equations put into a standard block tridiagonal form. Lower-upper decomposition ed to solve the resulting block matrix system with partial pivoting within the blocks to prevent the buildup of round-off errors. Reformulation of the governing equations became necessary with the discovery that the spline techniques, applied to the original formulation in Ref. [1], yielded a singular block matrix on the diagonal during forward substitution. By a slight recasting of the differential equation governing the pressure gradient, this singularity is removed in the matrix.

This study considers the solution of both self-similar and non-similar laminar boundary layer problems. Falkner-Skan problems with positive and negative wall shear and two non-similar flow situations are computed.

II. Analysis

Governing Differential Equations

The governing equations in this problem are the two-dimensional boundary layer equations for incompressible laminar flow. In dimensionless stream function form, the boundary layer equation is written

$$f_{\eta\eta\eta} + ff_{\eta\eta} + \beta(1 - f_{\eta}^{2}) = 2\xi(f_{\eta}f_{\xi\eta} - f_{\eta\eta}f_{\xi}) \qquad (1)$$

where

$$f_{\eta} = \frac{\partial f}{\partial \eta} \quad ; \quad f_{\xi} = \frac{\partial f}{\partial \xi} \quad .$$

The pressure gradient parameter, β , is given by

$$\beta(\xi) = \frac{2\xi}{u_e}\frac{du_e}{d\xi} \qquad (2)$$

where u_e is the velocity at the edge of the boundary layer. The boundary conditions for the computational domain, shown in Fig. 1, are identical to those given in Ref. [1].

$$f(\xi,0) = f_w(\xi) \qquad (3a)$$

$$f_{\eta}(\xi,0) = u_w(\xi) \qquad (3b)$$

$$f_{\eta}(\xi,\eta_{\infty}) = 1 \quad . \qquad (3c)$$

For the problem considered here, the mass transfer and velocity at the wall are taken to be zero. The quantity $\eta_{\infty} = \eta_{\infty}(\xi)$ is the location of the outer edge of the boundary layer, which, for simplicity, is taken as constant for this analysis. Equations (1) and (3) define the standard problem.

For the inverse problem, the wall shear is specified as

$$f_{\eta\eta}(\xi,0) = S(\xi) \qquad \xi > 0 \quad . \qquad (4)$$

Equations (1), (3) and (4) lead to an over-determined system. To complete the formulation of the mechul function scheme, the pressure gradient parameter is written as

$$\beta(\xi) = \beta(\xi,\eta) \quad .$$

Then in Ref. [1] the following derivative condition is introduced:

$$\frac{\partial \beta}{\partial \eta} = 0 \qquad \xi > 0 \quad . \tag{5}$$

The pressure gradient parameter is thus determined by solving Eqs. (1) and (5) together with boundary conditions (3) and (4).

For self-similar flows,

$$f_{\xi}(\xi,\eta) = 0 \quad . \tag{6}$$

Therefore, the boundary conditions and the pressure gradient are independent of the streamwise coordinate. In this case, the differential equations reduce to

$$f_{\eta\eta\eta} + f f_{\eta\eta} + \beta\left(1 - f_{\eta}^{2}\right) = 0 \tag{7a}$$

$$\beta_{\eta} = 0 \tag{7b}$$

yielding a system of ordinary differential equations.

It was found during the present study that the original gradient parameter condition, Eq. (5), used with the fourth order spline relations, leads to a singular matrix when solving the resulting block tridiagonal system by L-U decomposition. Using a constant normal stepsize in the spline $S^{1}(4,0)$ given in Ref. [3] yields a singular block matrix on the diagonal, thus preventing the solution of the matrix. This problem became apparent in the forward substitution step.

To eliminate this singularity, the pressure gradient parameter condition was changed to

$$\frac{\partial^2 \beta}{\partial \eta^2} = 0 \qquad \xi > 0 \quad . \tag{8}$$

Then, the condition

$$\frac{\partial \beta}{\partial \eta} = 0 \qquad \xi > 0 \tag{5}$$

is enforced at the wall with two point spline boundary conditions. Using the alternate fourth order spline relation $S^2(4,0)$ from Ref. [3] produces a non-singular matrix for constant or varying stepsize while still forcing the pressure gradient to be independent of η.

Differential Equations in First Order Form

The governing differential equations, (1), are written in first order form with the exception of Eq. (8):

$$f_\eta = u \tag{9a}$$

$$u_\eta = \tau \tag{9b}$$

$$\beta_{\eta\eta} = 0 \tag{9c}$$

$$\tau_\eta = \beta(1 - u^2) - f\tau + 2\xi(uu_\xi - \tau f_\xi) \quad . \tag{9d}$$

Then the boundary conditions are

$$f(\xi,0) = 0 \tag{10a}$$

$$u(\xi,0) = 0 \tag{10b}$$

$$\tau(\xi,0) = S(\xi) \tag{10c}$$

$$u(\xi,\eta_\infty) = 1 \quad . \tag{10d}$$

Streamwise Discretization

The governing differential equations are discretized using fourth order accurate splines in the normal direction and backward differences in the streamwise direction. For the ξ derivatives, the following general backward difference formula for constant stepsize is used.

$$(g_\xi)_{i,j} = \frac{1}{\Delta\xi}\left[ag_{i,j} + bg_{i-1,j} + cg_{i-1,j}\right] \quad . \tag{11}$$

For $i > 2$, the three point second-order accurate version is used with

$$a = \frac{3}{2} \quad , \quad b = -1 \quad , \text{ and } c = \frac{1}{2} \tag{11a}$$

whereas, for the second downstream ξ station ($i = 2$), the first-order accurate (two point) formula is used with

$$a = 1 \quad , \quad b = -1 \quad , \text{ and } c = 0 \quad . \tag{11b}$$

Examining the governing equations, only Eq. (9d) contains streamwise derivaties. Applying Eq. (11) to the u_ξ and f_ξ derivatives yields

$$\begin{aligned}(\tau_\eta)_{i,j} &= \beta_{i,j}\left(u_{i,j}^2 - 1\right) - f_{i,j}\tau_{i,j} \\ &+ \alpha_{i,j}\left(au_{i,j}^2 + bu_{i-1,j}^2 + cu_{i-1,j}^2\right) \\ &- 2\alpha_{i,j}\left(af_{i,j}\tau_{i,j} + bf_{i-1,j}\tau_{i,j} + cf_{i-2,j}\tau_{i,j}\right)\end{aligned} \tag{12}$$

where

$$\alpha_{i,j} = \frac{\xi_i}{\Delta\xi_i} \quad .$$

Normal Discretization

Since first and second derivatives with respect to η occur in the equations, the splines $S^1(4,0)$ and $S^2(4,0)$ are used. To apply the spline approximations in the normal direction, the following spline derivatives are defined:

$$\ell^f = f_\eta \tag{13a}$$

$$\ell^u = u_\eta \tag{13b}$$

$$\ell^\tau = \tau_\eta \tag{13c}$$

$$\ell^\beta = \beta_\eta \tag{13d}$$

$$L^\beta = \beta_{\eta\eta} \quad . \tag{13e}$$

These definitions are then substituted into the governing equations.

Substituting definitions (13) into Eqs. (9) yields

$$\ell^f_{i,j} = u_{i,j} \tag{14a}$$

$$\ell^u_{i,j} = \tau_{i,j} \tag{14b}$$

$$L^{\beta}_{i,j} = 0 \tag{14c}$$

$$\ell^{\tau}_{i,j} = \beta_{i,j}\left(u^2_{i,j} - 1\right) - \left(2a\alpha_{i,j} + 1\right)f_{i,j}\tau_{i,j}$$

$$- 2\alpha_{i,j}\left(bf_{i-1,j} + cf_{i-2,j}\right)\tau_{i,j}$$

$$+ \alpha_{i,j}\left(au^2_{i,j} + bu^2_{i-1,j} + cu^2_{i-2,j}\right) \quad . \tag{14d}$$

From Ref. [3], the tridiagonal relation for $S^1(4,0)$ is, at a streamwise station i,

$$\ell^{g}_{j+1} + (1 + \sigma)^2\ell^{g}_{j} + \sigma^2\ell^{g}_{j-1} =$$

$$\frac{2}{1 + \sigma}\frac{1}{h_j}\left[\frac{1 + 2\sigma}{\sigma}\, g_{j+1} + \frac{\sigma - 1}{\sigma}\,(1 + \sigma)^3 g_j\right.$$

$$\left. - \sigma^2(2 + \sigma)g_{j-1}\right]$$

where

$$h_j = \Delta\eta_j \quad ,$$

$$\sigma = \sigma_j = \frac{h_{j+1}}{h_j} \quad ,$$

and ℓ_j^g is the first derivative spline approximation of $(\partial g/\partial \eta)_j$. The tridiagonal relation for $S^2(4,0)$ is given as

$$\frac{\sigma^2 - \sigma - 1}{12\sigma} L_{j+1}^g + \frac{\sigma^3 + 4\sigma^2 + 4\sigma + 1}{12\sigma} L_j^g + \frac{1 + \sigma - \sigma^2}{12} L_{j-1}^g$$

$$= \frac{1}{h_j^2} \left[\frac{g_{j+1}}{\sigma} - \frac{1 + \sigma}{\sigma} g_j + g_{j-1}\right] \quad . \tag{16}$$

where L_j^g is the second derivative spline approximation of $(\partial^2 g/\partial \eta^2)_j$.

To eliminate the spline derivatives, Eqs. (14a), (14b) and (14d) are substituted into the $S^1(4,0)$ relation given by Eq. (15). The second derivative condition, Eq. (14c), can be substituted into the $S^2(4,0)$ relation. This yields the following four tridiagonal relations (i subscript understood). All coefficients are given in Appendix A.

$$\sigma 2 u_{j-1} + \sigma 1 u_j + u_{j+1} = \hat{\sigma} 3 f_{j-1} + \hat{\sigma} 4 f_j + \hat{\sigma} 5 f_{j+1} \tag{17a}$$

$$\sigma 2 \tau_{j-1} + \sigma 1 \tau_j + \tau_{j+1} = \hat{\sigma} 3 u_{j-1} + \hat{\sigma} 4 u_j + \hat{\sigma} 5 u_{j+1} \tag{17b}$$

$$\hat{\sigma} 8 \beta_{j-1} + \hat{\sigma} 7 \beta_j + \hat{\sigma} 6 \beta_{j+1} = 0 \tag{17c}$$

$$\sigma 2\left[(c1 + \beta)u^2 - \beta + c2f\tau + \overline{E1_j}\tau\right]_{j-1}$$

$$+ \sigma 1\left[(c1 + \beta)u^2 - \beta + c2f\tau + \overline{E2_j}\tau\right]_{j}$$

$$+ \left[(c1 + \beta)u^2 - \beta + c2f\tau + \overline{E3_j}\tau\right]_{j+1}$$

$$= - \sigma 2C4_{j-1} - \sigma 1c4_j - c4_{j+1} \quad . \tag{17d}$$

Linearization

The block tridiagonal relations given by Eq. (17) form a nonlinear system relating $(f,u,\tau,\beta)_{i,j}$. These equations are first linearized and then solved using Newton's method. The Newton iterates are given by

$$f_{i,j}^{(n+1)} = f_{i,j}^{(n)} + \delta f_{i,j}^{(n)} \tag{18a}$$

$$u_{i,j}^{(n+1)} = u_{i,j}^{(n)} + \delta u_{i,j}^{(n)} \tag{18b}$$

$$\tau_{i,j}^{(n+1)} = \tau_{i,j}^{(n)} + \delta \tau_{i,j}^{(n)} \tag{18c}$$

$$\beta_{i,j}^{(n+1)} = \beta_{i,j}^{(n)} + \delta \beta_{i,j}^{(n)} \tag{18d}$$

where the superscript indicates the iteration number. Equations (18) are substituted into the tridiagonal relations at the $(n+1)^{st}$ iteration and the quadratic and higher order terms are neglected. This allows the

equations to be solved for the unknown corrections, with all other terms being known at iteration (n). Moving known quantities to the right hand side, the linearized correction equations in block tridiagonal form at streamwise station i are

$$-\hat{\sigma}3\delta f_{j-1} + \sigma 2\delta u_{j-1} - \hat{\sigma}4\delta f_j + \sigma 1\delta u_j - \hat{\sigma}5\delta f_{j+1} + \delta u_{j+1} = L_j \qquad (19a)$$

$$-\hat{\sigma}3\delta u_{j-1} + \sigma 2\delta \tau_{j-1} - \hat{\sigma}4\delta u_j + \sigma 1\delta \tau_j - \hat{\sigma}5\delta u_{j+1} + \delta \tau_{j+1} = P_j \qquad (19b)$$

$$-\hat{\sigma}8\delta\beta_{j-1} - \hat{\sigma}7\delta\beta_j - \hat{\sigma}6\delta\beta_{j+1} = Q_j \qquad (19c)$$

$$\sigma 2\left[\hat{A}\delta f + \hat{B}\delta u + \hat{D}_j\delta\tau + \hat{G}\delta\beta\right]_{j-1} + \sigma 1\left[\hat{A}\delta f + \hat{B}\delta u + \hat{E}_j\delta\tau + \hat{G}\delta\beta\right]_j$$

$$+ \left[\hat{A}\delta f + \hat{B}\delta u + \hat{F}_j\delta\tau + \hat{G}\delta\beta\right]_{j+1} = T_j \qquad (19d)$$

where $\hat{A}_j$, $\hat{B}_j$, $\hat{D}_j$, $\hat{E}_j$ and $\hat{F}_j$ are coefficients obtained from the linearization of the equations (refer to Appendix A).

These block tridiagonal equations can then be written in the following matrix form (i subscript understood)

$$B_jZ_{j-1} + A_jZ_j + C_jZ_{j+1} = R_j \quad , \quad 2 \leq j \leq N \qquad (20)$$

where

$$Z_j = \begin{bmatrix} \delta f \\ \delta u \\ \delta \tau \\ \delta \beta \end{bmatrix}_j \qquad (21)$$

and A_j, B_j, C_j are 4×4 block matrices and R_j is the column vector of known quantities.

Two Point Spline Relations at the Wall

The boundary conditions at the wall in correction form, at streamwise station i, are

$$\delta f_1 = 0 \tag{22a}$$

$$\delta u_1 = 0 \tag{22b}$$

$$\delta \tau_1 = 0 \quad . \tag{22c}$$

One additional condition must be provided to close the system at the wall. A two point spline boundary condition, given in Ref. [3], is used.

$$\beta_2 - \beta_1 - \frac{h_2}{2}\left(\ell_2^\beta - \ell_1^\beta\right) + \frac{h_2^2}{12}\left(L_2^\beta - L_1^\beta\right) = 0 \quad . \tag{23}$$

This relation allows the original pressure gradient condition to be enforced at the wall. From the governing equations,

$$L_j^\beta = 0$$

and, enforcing the original gradient condition

$$\ell_j^\beta = 0 \quad ,$$

yields the simple relationship

$$\beta_2 = \beta_1 \tag{24a}$$

or, in correction form

$$\delta\beta_2 = \delta\beta_1 \quad . \tag{24b}$$

The linearized block system for the solution corrections at the wall can then be written in the following matrix form:

$$A_1 Z_1 + C_1 Z_2 = R_1 \tag{25}$$

with Z_j defined by Eq. (21).

Two Point Spline Relation at the Outer Edge

The far field boundary condition is, at any i,

$$\delta u_{N+1} = 0 \quad . \tag{26}$$

Three additional conditions are required to close the system. The two point spline conditions are again applied. For one relation, the following is used:

$$f_{N+1} - f_N - \frac{h_{N+1}}{2}\left(u_{N+1} + u_N\right) + \frac{h_{N+1}^2}{12}\left(\tau_{N+1} - \tau_N\right) = 0 \quad . \tag{27a}$$

Differentiating Eq. (27a) with respect to η yields

$$u_{N+1} - u_N - \frac{h_{N+1}}{2}\left(\tau_{N+1} + \tau_N\right) + \frac{h_{N+1}^2}{12}\left(\ell_{N+1}^{\tau} - \ell_N^{\tau}\right) = 0 \quad . \tag{27b}$$

Differentiating once more gives

$$\tau_{N+1} - \tau_N - \frac{h_{N+1}}{2}\left(\ell_{N+1}^{\tau} + \ell_N^{\tau}\right) + \frac{h_{N+1}^2}{12}\left(L_{N+1}^{\tau} - L_N^{\tau}\right) = 0 \quad .$$

Assuming the difference between the second order terms to be negligible compared to the lower order terms, the above equation simplifies to

$$\tau_{N+1} - \tau_N - \frac{\Delta h_{N+1}}{2}\left(\ell^{\tau}_{N+1} + \ell^{\tau}_{N}\right) = 0 \quad . \tag{27c}$$

Equations (27) provide the three additional conditions required for the far field. These equations are linearized and solved for the correction terms as before and then written in matrix form.

$$B_{N+1} Z_N + A_{N+1} Z_{N+1} = R_{N+1} \quad . \tag{28}$$

Solution of Block Tridiagonal Matrix Equation

The block tridiagonal system formed by Eqs. (20), (25) and (28) is solved using standard lower-upper (LU) decomposition together with partial pivoting within the 4 × 4 blocks to prevent the buildup of roundoff errors. A block tridiagonal solver using subroutines developed by Blottner [4] which perform the partial pivoting is used to solve the matrix equations for the corrections at each iteration.

Starting Solution

The Falkner-Skan self-similar equations are obtained by setting ξ equal to zero in Eqs. (1) and (8).

$$f_{\eta\eta\eta} + f f_{\eta\eta} + \beta\left(1 - f_{\eta}^2\right) = 0 \tag{29a}$$

$$\beta_{\eta\eta} = 0 \quad . \tag{29b}$$

The boundary conditions given by Eqs. (3) and (4), with no dependence on ξ, are used. The solution of this ordinary differential equation with splines

approximating all η derivatives is used as the starting solution for the non-similar case. The scheme is derived so that the Falkner-Skan solutions for positive and negative wall shear can also be computed.

As an initial guess for the starting solution, a fourth order Pohlhausen-type polynomial is used to approximate $(f,u,\tau,\beta)_{i,j}$ at $\xi = 0$. The Pohlhausen polynomial for the velocity is of the form

$$u = b\zeta + c\zeta^2 + d\zeta^3 + e\zeta^4 \tag{30}$$

where

$$\zeta = \frac{\eta}{\eta_\infty} \quad , \quad u = u(\eta) \quad ,$$

and the constants b, c, d and e can be found by applying the boundary conditions and the ordinary differential equation. The stream function and the shear can be found by integrating and differentiating Eq. (30) respectively. By substituting into the differential equation, Eq. (29a), an approximation for β is obtained in the form

$$\beta = \frac{6S\eta_\infty - 12}{\eta_\infty^2} \quad . \tag{31}$$

Once the starting solution is determined, the second streamwise station must be treated in a special way for the non-similar flow case. With only one previous streamwise step known, two point backward differences are used to approximate the ξ derivatives. Past this station, three point backward differences are used. The solution profile at the last calculated station is used as the first approximation at the new station.

III. Results and Discussion

Similar Flow Problem

Computations for similar flows were performed for positive and negative wall shears. For all positive wall shears, the solutions were obtained independently for a specific shear. For the negative wall shears, the sensitivity of the method to the initial guess required calculating solutions consecutively for small steps in shear and using the previous solution profile as an initial guess for the next profile. The solution process was begun for a zero wall shear, $S = 0$, and the shear was decremented by 0.01 or 0.05.

Solution comparisons are made between the present reformulated scheme, the original mechul function formulation [1], and the nonlinear eigenvalue scheme [2]. This method was developed by Keller and Cebeci before the mechul function scheme. The eigenvalue method solves the inverse problem by treating the unknown pressure gradient as an eigenvalue. Then, two iteration procedures, an "inner" and an "outer" iteration, are performed. The inner iteration solves the governing equations for a standard problem assuming β is known. This inner iteration is then used with Newton's method to determine the pressure gradient parameter using the variational equations in the outer iteration procedure. The variational equations are the standard boundary layer equations differentiated with respect to β.

For positive wall shears computed with the reformulated scheme, a normal stepsize of $\Delta\eta = 0.15$ is used with $\eta_\infty = 6$; for reverse flow solutions and separation ($S = 0$), $\Delta\eta = 0.15$ and $\eta_\infty = 9$ are used. The criteria used for convergence is

$$\varepsilon = \left|\beta^{(n+1)} - \beta^{(n)}\right| < 10^{-8}$$

Cebeci and Keller applied a similar convergence test to the original scheme as well as the eigenvalue scheme with $\varepsilon \leq 10^{-4}$ [1,2].

The results of the self-similar flow calculations with positive wall shear are given in Table I. These results are compared with those of Smith [5]. Comparison shows the values from the reformulated mechul function scheme closely approach those of Smith. All cases converged quadratically. It should be noted that the greater number of iterations required for convergence with the reformulated scheme are the result of the present more severe convergence criteria.

The results of the reverse flow computations are presented in Table II. These results are compared with those of Stewartson [6]. Again, the agreement is good. Here, the number of iterations decreases appreciably with the use of consecutive calculations. Figure 2 shows examples of both positive and negative wall shear velocity profiles.

Non-Similar Flow Problem

Non-similar flow computations were performed using two linear wall shear distributions. The first case, given by

$$S(\xi) = 0.4696\ (1 - \xi)$$

has a zero pressure gradient at $\xi = 0$, indicating a flat plate flow. The second case, given by

$$S(\xi) = 1.23259\ (1 - \xi)$$

has a pressure gradient parameter of unity at $\xi = 0$, indicating a stagnation point. Both cases approach zero shear at $\xi = 1$, yielding separation. Both cases were computed with values of $\eta_\infty = 6$ and $\eta_\infty = 9$. For all non-similar computations, a streamwise stepsize of $\Delta\xi = 0.05$ was used, with $\Delta\eta = 0.25$.

The convergence criteria applied is identical to that used for the self-similar cases.

A comparison of results for the first case is given in Tables III and IV. Table III compares the results with those of Refs. [1] and [2]. The agreement with the Richardson extrapolation results obtained from the eigenvalue scheme values is quite good. The numerical instability experienced by the original formulation does not appear in the present method. It was found that with $\eta_\infty = 6$, the solution would not converge at ξ stations near separation, yet with $\eta_\infty = 9$, the solution would march through the separation point at $\xi = 1$. Once past separation, however, the scheme quickly becomes unstable and the solution does not converge. Table IV compares results for Case 1 with $\eta_\infty = 6$ and $\eta_\infty = 9$.

A comparison of results for the second case is given in Tables V and VI. In Table V, the results are compared with results from Ref. [2], since no results are available for this case with the original mechul function formulation. The agreement with the Richardson extrapolation is again very good. As with case one, the results could be obtained through separation only with $\eta_\infty = 9$. Table VI compares the results for the two values of η_∞.

Figures 3 and 4 are plots the pressure gradient parameter as a function of ξ for both non-similar cases. The results are smooth, with a minimum occurring just before separation for computations performed with $\eta_\infty = 9$.

A Note on Normal Stepsize

For all numerical cases presented, the normal stepsize, $\Delta\eta$, is kept constant for reasons of comparison with Refs. [1] and [2]. Non-similar cases were computed, however, using a geometric progression for η, with $\eta_{j+1}/\eta_j = 1.1$. Using this geometric progression, both non-similar cases

would proceed through separation with $\eta_\infty = 6$. The results obtained are identical with those produced with a constant stepsize and $\eta_\infty = 9$. Using the geometric progression also allowed the number of normal steps to be halved without decreasing solution accuracy.

IV. Conclusion

An accurate, stable, and efficient method is developed to determine the pressure gradient distribution on a body surface in a laminar boundary layer flow with wall shear specified. The reformulated mechul function scheme presented here does not exhibit the numerical instability experienced with non-similar type flow problems in Ref. [1]. The method is fully implicit and is applicable to Falkner-Skan as well as non-similar flow problems. Reverse flow self-similar problems have also been computed but are found to be very sensitive to the initial solution guess.

The reformulated scheme uses fourth order splines to approximate derivatives in the normal direction and three point backward differences for the streamwise derivatives, both to aid solution stability. Partial pivoting is used within the 4×4 blocks resulting from the discretized, linearized equations of motion. These modifications prevent the instability described in Ref. [1].

The results for both similar and non-similar cases were found to be stable and accurate with quadratic convergence consistently observed. It was found that solution calculations would proceed through the separation point with constant stepsize and $\eta_\infty = 9$, or with a geometric progression and $\eta_\infty = 6$. Although all solutions converged quadratically upstream of the separation point, using a more accurate initial guess would decrease iteration counts further.

References

(1) Cebeci, T. and H. B. Keller, "Laminar Boundary Layers with Assigned Wall Shear," Proceedings of the Third International Conference on Numerical Methods in Fluid Dynamics, Paris, France (July 1972); Lecture Notes in Physics, Vol. 19, pp. 79-85, Springer-Verlag (1973).

(2) Keller, H. B. and T. Cebeci, "An Inverse Problem in Boundary Layer Flows: Numerical Determination of Pressure Gradient for a Given Wall Shear," Jour. Comp. Phys. 10:151 (1972).

(3) Rubin, S. G. and P. K. Khosla, "Polynomial Interpolation Methods for Viscous Flow Calculations," Jour. Comp. Phys. 24:217 (1977).

(4) Blottner, F. G., "Introduction to Computational Methods for Boundary Layers," Sandia Report No. 79-0893.

(5) Smith, A. M. O., "Improved Solutions of the Falkner-Skan Boundary Layer Equation," Sherman M. Fairchild Fund Paper No. FF-10 (1954).

(6) Stewartson, K., "Further Solutions of the Falkner-Skan Equation," Cambridge Phil. Soc. 50:454-465 (1954).

(7) Ahlberg, G. H., E. N. Nilson and J. L. Walsh, The Theory of Splines and Their Applications, Academic Press, New York (1967).

(8) Cebeci, T. and P. Bradshaw, Momentum Transfer in Boundary Layers, Hemisphere Publishing Corp. (1977).

(9) Evans, H. L., Laminar Boundary Layer Theory, Addison Wesley Publishing Co., Menlo Park, CA (1966).

(10) Rosenhead, L., Laminar Boundary Layers, Oxford University Press, Oxford, U.K. (1963).

(11) Smith, G. D., Numerical Solution of Partial Differential Equations: Finite Difference Methods, Oxford University Press, Oxford, U.K. (1978).

(12) White, F. M., Viscous Fluid Flow, McGraw-Hill Inc., New York (1974).

(13) Atkinson, K. E., An Introduction to Numerical Analysis, John Wiley and Sons, Inc., New York (1978).

(14) Schlichting, H., Boundary Layer Theory, 7th Edition, McGraw-Hill Inc., New York (1979).

TABLE I

COMPARISON OF POSITIVE WALL SHEAR SOLUTIONS FOR SELF-SIMILAR FLOWS ($\nu \equiv$ ITERATION NUMBER)

$\eta_\infty = 6$

f_w''	REFORMULATED MECHUL FUNCTION		CEBECI & KELLER MECHUL FUNCTION		NONLINEAR EIGEN -VALUE SCHEME		SMITH
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
.40032	0.050002	4	0.05025	3	0.05031	3	0.05
.31927	0.100001	4	0.10021	3	0.10017	3	0.10
.23974	0.140003	4	0.14019	3	0.14024	3	0.14
.19078	0.160008	5	0.16019	3	0.16016	3	0.16
.12864	0.180019	5	0.18020	3	0.18025	3	0.18
.05517	0.195057	6	0.19524	3	0.19528	3	0.195
0	0.198837	5	0.19917	3	0.20259	2	0.198834

4 May 1984
KCK:GHH:lhz

TABLE II

COMPARISON OF REVERSE-FLOW SOLUTIONS FOR SELF-SIMILAR FLOWS.

$$\eta_\infty = 9$$

f''_w	REFORMULATED MECHUL FUNCTION		CEBECI& KELLER MECHUL FUNCTION		NONLINEAR EIGEN-VALUE SCHEME		STEWARTSON
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
-0.097	0.180340	3	0.18074	4	0.18143	2	0.18
-0.132	0.152116	4	0.15234	4	0.15416	6	0.15
-0.141	0.132974	4	0.13412	4	0.13545	6	—

4 May 1984
KCK:GHH:lhz

TABLE III

COMPUTED PRESSURE GRADIENT PARAMETER β AS A FUNCTION OF ξ FOR THE WALL SHEAR DISTRIBUTION:

$$S(\xi) = 0.4696\,(1-\xi)$$

$$\eta_\infty = 6$$

ξ	REFORMULATED MECHUL FUNCTION		CEBECI & KELLER MECHUL FUNCTION		NONLINEAR EIGEN-VALUE SCHEME		EIGENVALUE SCHEME WITH RICHARDSON EXTRAPOL.
	$-\beta$	ν	$-\beta$	ν	$-\beta$	ν	$-\beta$
0	-0.000004	4	0.00179	3	0.00179	3	0.00003
.10	0.043781	4	0.04535	3	0.04532	4	0.04383
.20	0.084248	4	0.08559	3	0.08553	5	0.09068
.30	0.121154	4	0.12231	3	0.12225	5	0.12113
.40	0.154257	4	0.15529	3	0.15522	5	0.15018
.50	0.183241	4	0.18420	3	0.18408	5	0.18308
.60	0.207657	4	0.20857	3	0.20845	5	0.20747
.70	0.226840	4	0.22685	3	0.22761	4	0.22664
.80	0.239734	5	0.22378	3	0.24041	4	0.23940
.90	0.251269	8	0.16835	3	0.24479	3	0.24376

4 May 1984
KCK:CHH:liz

TABLE IV

COMPARISON OF β AS A FUNCTION OF ξ WITH VARYING η_∞, FOR THE WALL SHEAR DISTRIBUTION:

$$S(\xi) = 0.4696\ (1-\xi)$$

ξ	MECHUL FUNCTION WITH $\eta_\infty = 6$		MECHUL FUNCTION WITH $\eta_\infty = 9$	
	$-\beta$	ν	$-\beta$	ν
0	-0.000004	4	-0.000004	4
.10	0.043781	4	0.043780	4
.20	0.084248	4	0.084247	4
.30	0.121154	4	0.121152	4
.40	0.154257	4	0.154254	4
.50	0.183241	4	0.183234	4
.60	0.207657	4	0.207637	4
.70	0.226840	4	0.226777	4
.80	0.239734	5	0.239550	4
.90	0.251269	8	0.243963	4
1.00	—	—	0.235625	5
1.10	—	—	0.207213	7

4 May 1984
KCK:GHH:lhz

TABLE V

COMPUTED PRESSURE GRADIENT β AS A FUNCTION OF ξ FOR THE WALL SHEAR DISTRIBUTION:

$$S(\xi) = 1.23259\ (1-\xi)$$

$$\eta_\infty = 6$$

ξ	REFORMULATED MECHUL FUNCTION		NONLINEAR EIGENVALUE SCHEME	EIGENVALUE SCHEME W/ RICHARDSON EXTRAPOLATION
	β	ν	β	β
0	0.999993	4	0.98862	0.99938
.10	0.760840	4	0.75308	0.76053
.20	0.542190	4	0.53733	0.54213
.30	0.344378	4	0.34164	0.34445
.40	0.167607	4	0.16634	0.16772
.50	0.012386	4	0.01203	0.01248
.60	-0.120291	4	-0.12017	-0.12017
.70	-0.228585	4	-0.22837	-0.22844
.80	-0.308985	4	-0.30877	-0.30856
.90	-0.352992	7	-0.35302	-0.35232
.95	-0.366964	10	-0.45517	-0.35427

4 May 1984
KCK:CHH:lhz

TABLE VI

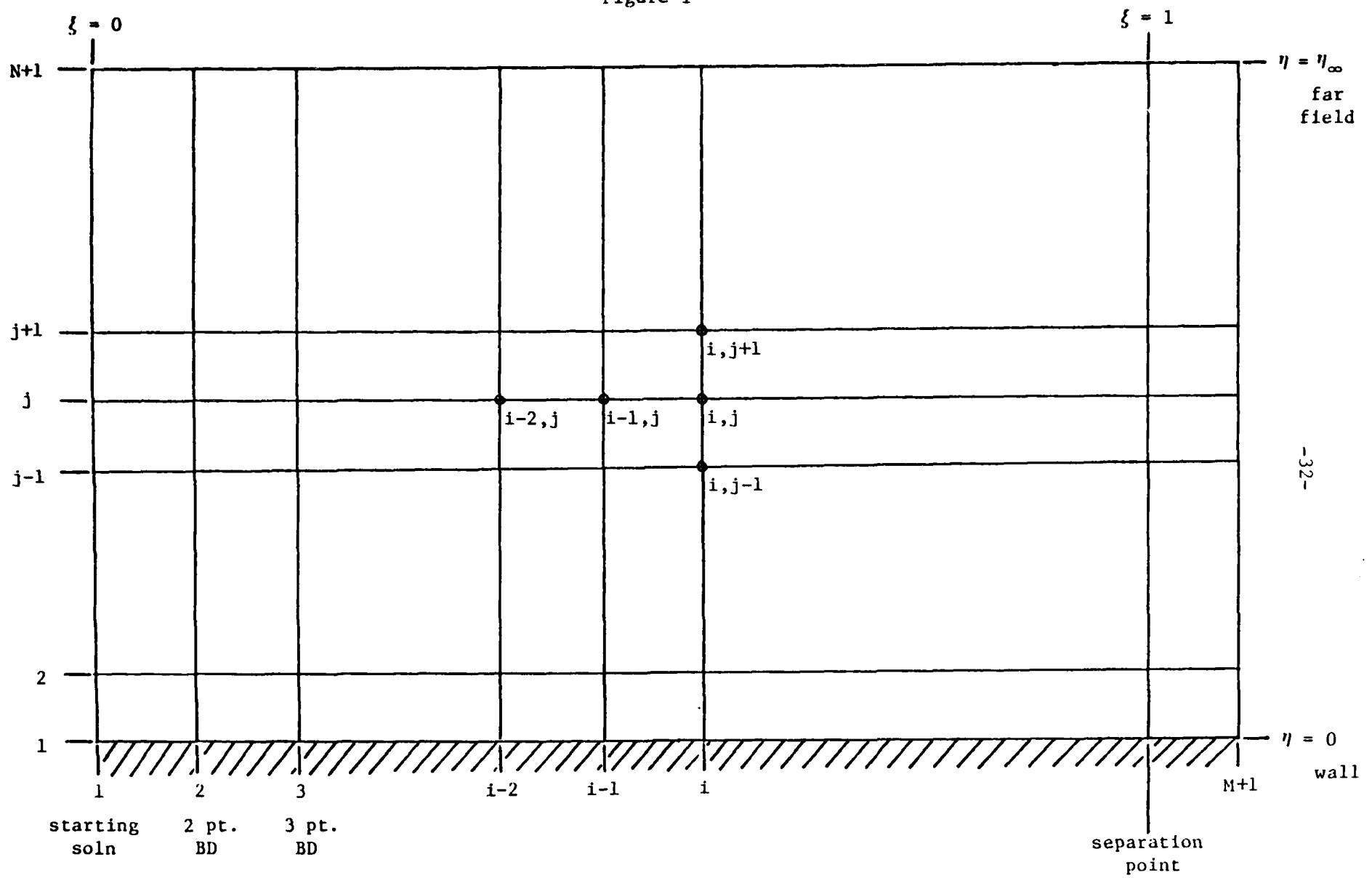
COMPARISON OF β AS A FUNCTION OF ξ WITH VARYING η_∞, FOR THE WALL SHEAR DISTRIBUTION:

$$S(\xi) = 1.23259\ (1-\xi)$$

ξ	MECHUL FUNCTION WITH $\eta_\infty = 6$		MECHUL FUNCTION WITH $\eta_\infty = 9$	
	β	ν	β	ν
0	0.999993	4	0.999993	4
.10	0.760840	4	0.760840	4
.20	0.542190	4	0.542190	4
.30	0.344378	4	0.344378	4
.40	0.167607	4	0.167607	4
.50	0.012386	4	0.012386	4
.60	-0.120291	4	-0.120289	4
.70	-0.228585	4	-0.228577	4
.80	-0.308985	4	-0.308911	4
.90	-0.352992	7	-0.353486	4
1.00	—		-0.339311	5

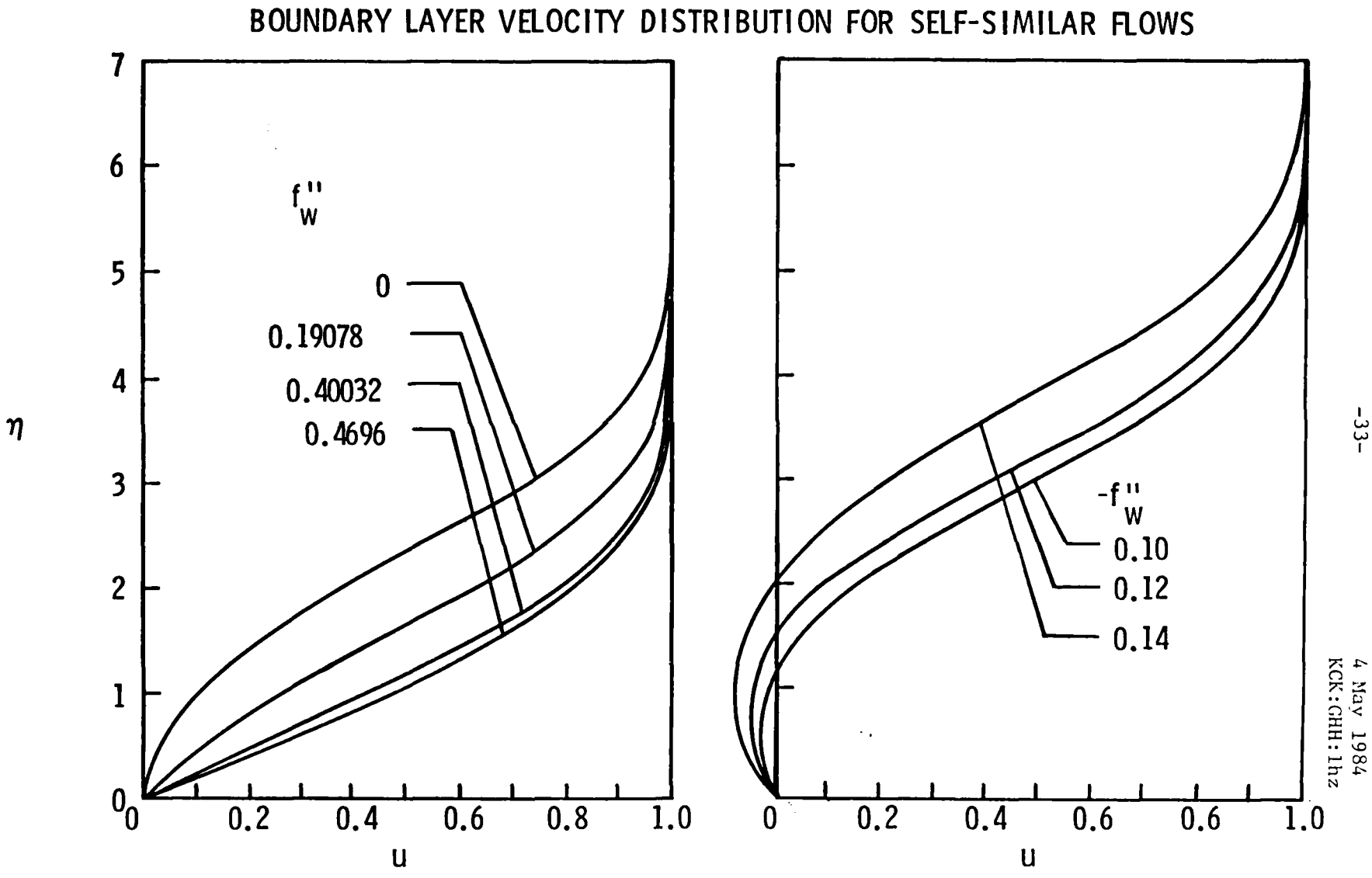
4 May 1984
KCK:GHH:llz

Figure 1



Computational Domain

Figure 2



4 May 1984
KCK:GHH:lhz

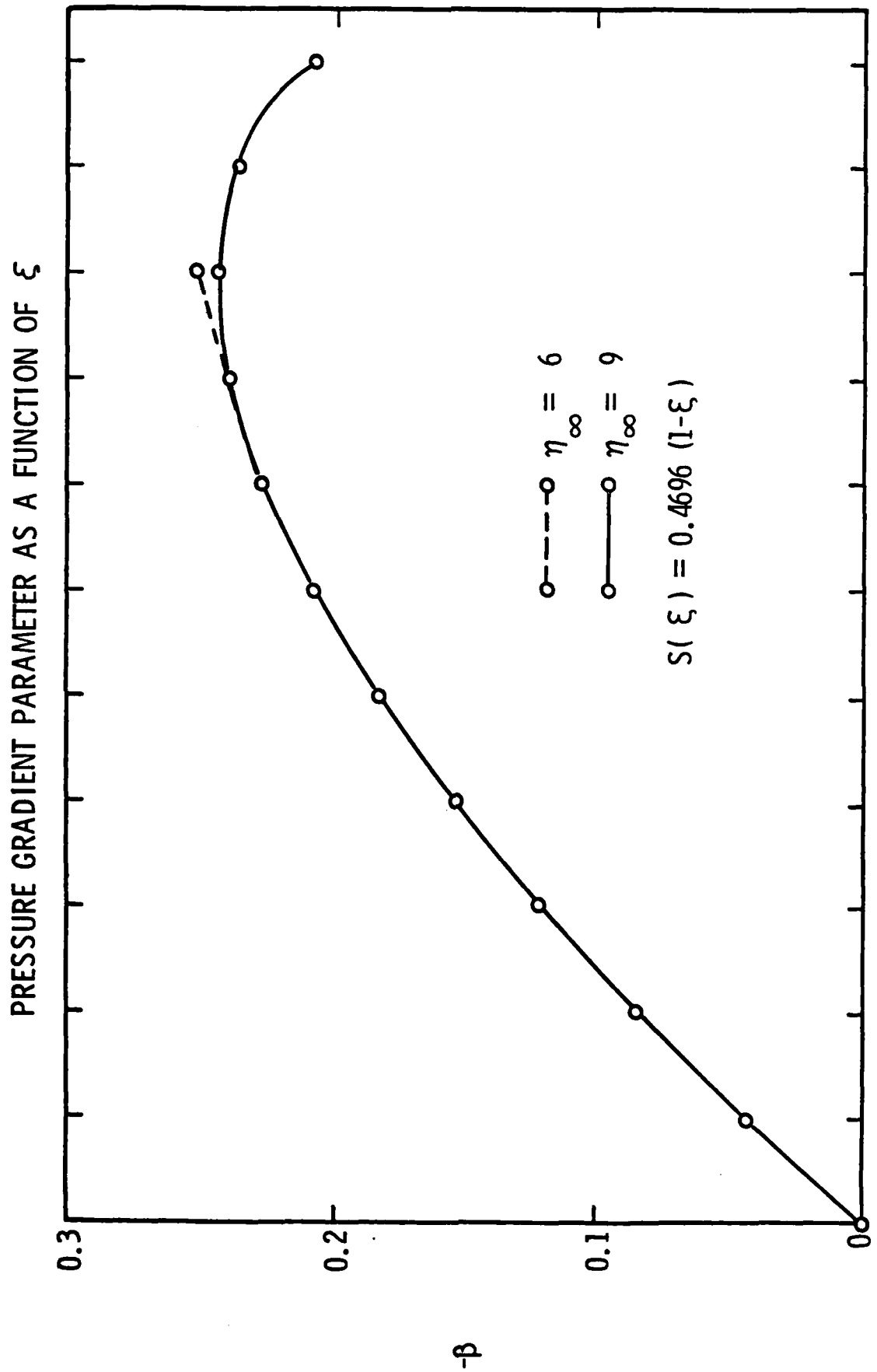
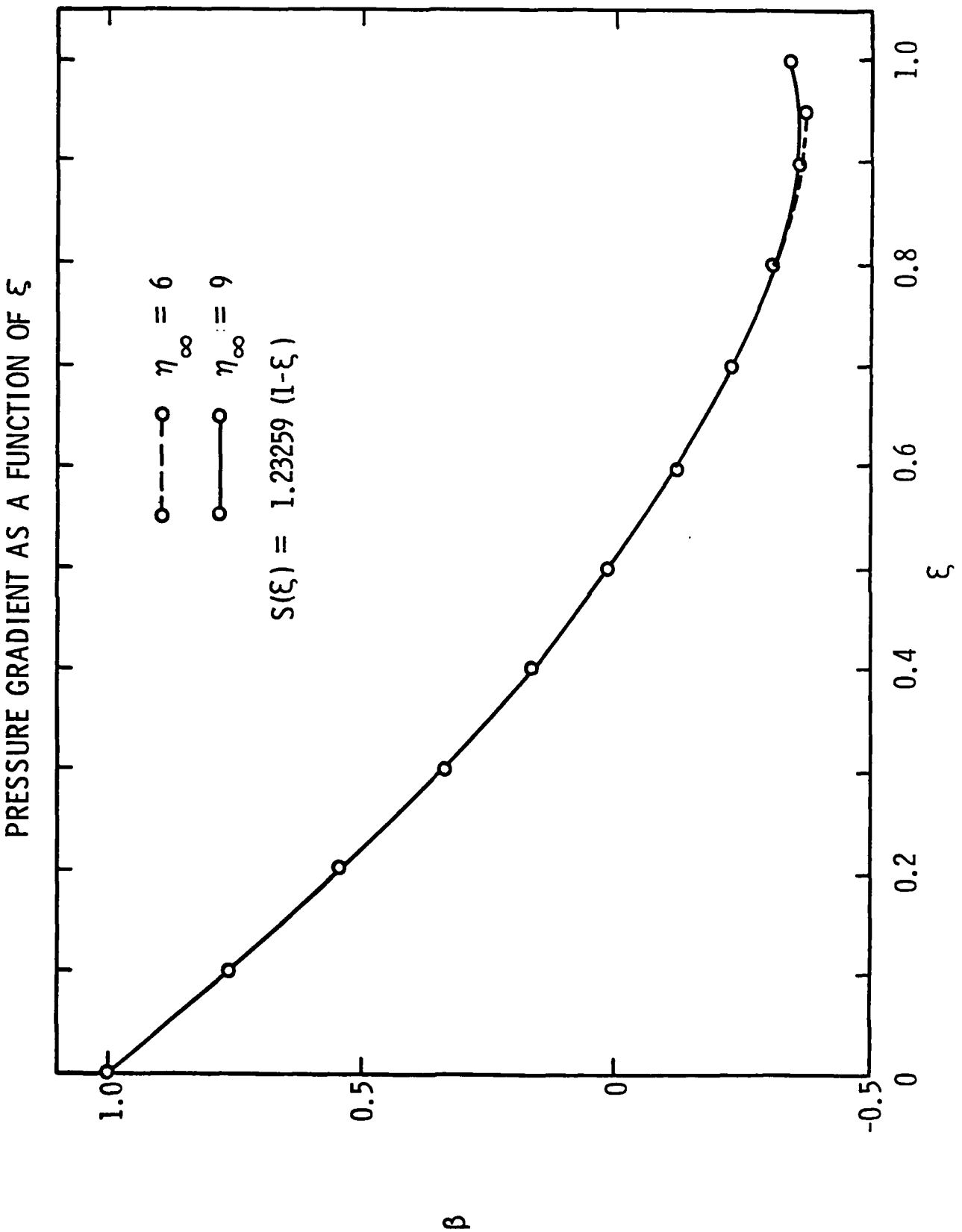

PRESSURE GRADIENT PARAMETER AS A FUNCTION OF ξ
-β
ξ
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1.0
η_∞ = 6
η_∞ = 9
S(ξ) = 0.4696 (1-ξ)

Figure 3

Figure 4


PRESSURE GRADIENT AS A FUNCTION OF ξ
$\eta_\infty = 6$
$\eta_\infty = 9$
$S(\xi) = 1.23259\ (1-\xi)$
β
ξ
1.0
0.5
0
-0.5
0
0.2
0.4
0.6
0.8
1.0

Appendix A

Fourth Order Spline Coefficients:

$$\sigma 1 = (\sigma + 1)^2$$

$$\sigma 2 = \sigma^2$$

$$\hat{\sigma}3 = -\frac{1}{h_j}\frac{2\sigma^2(2+\sigma)}{1+\sigma}$$

$$\hat{\sigma}4 = \frac{1}{h_j}\frac{2(\sigma-1)(\sigma+1)}{\sigma}$$

$$\hat{\sigma}5 = \frac{1}{h_j}\frac{2(1+2\sigma)}{1+\sigma}$$

$$\hat{\sigma}6 = \frac{1}{h_j^2}\frac{1}{\sigma}$$

$$\hat{\sigma}7 = -\frac{1}{h_j^2}\frac{1+\sigma}{\sigma}$$

$$\hat{\sigma}8 = \frac{1}{h_j^2}$$

Streamwise Discretization Coefficients:

$$\alpha_{i,j} = \frac{\xi_i}{\Delta\xi_i}$$

$$c1_j = a\alpha_i$$

$$c2_j = -(2a\alpha_i + 1)$$

$$c3_j = -2(b\alpha_i f_{i-1,j} + c\alpha_i f_{i-2,j})$$

$$c4_j = \alpha_i(bu^2_{i-1,j} + cu^2_{i-2,j})$$

$$\overline{E1_j} = C3_{j-1} - \frac{\hat{\sigma}3}{\sigma 2}$$

$$\overline{E2_j} = C3_j - \frac{\hat{\sigma}4}{\sigma 1}$$

$$\overline{E3_j} = C3_{j+1} - \hat{\sigma}5$$

Coefficients Resulting from Linearization:

$$\hat{A}_j = C2_j I_j$$

$$\hat{B}_j = 2u_j(C1_j + \beta_j)$$

$$\hat{C}_j = C2_j f_j$$

$$\hat{D}_j = \hat{C}_{j-1} + \overline{E1_j}$$

$$\hat{E}_j = \hat{C}_j + \overline{E2_j}$$

$$\hat{F}_j = \hat{C}_{j+1} + \overline{E3_j}$$

$$\hat{G}_j = u_j^2 - 1$$

DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL MEMORANDUM FILE NO. 84-79,
by K. C. Kaufman and G. H. Hoffman, dated 4 May 1984

Office of Naval Research
Department of the Navy
800 North Quincy Street
Arlington, VA 22217
Attn: R. E. Whitehead
(Copy No. 1)

Office of Naval Research
Department of the Navy
800 North Quincy Street
Arlington, VA 22217
Attn: C. Lee
(Copy No. 2)

Commander
Naval Sea Systems Command
Department of the Navy
Washington, DC 20362
Attn: T. E. Peirce
Code NSEA 63R31
(Copy No. 3)

Commander
Naval Underwater Systems Center
Department of the Navy
Newport, RI 02840
Attn: D. J. Goodrich
Code 3634
(Copy No. 4)

Commander
David W. Taylor Naval Ship
Research & Development Center
Department of the Navy
Bethesda, MD 20084
Attn: Library
(Copy No. 5)

Commander
David W. Taylor Naval Ship
Research & Development Center
Department of the Navy
Bethesda, MD 20084
Attn: Y. T. Lee
Code 1542
(Copy No. 6)

Commander
David W. Taylor Naval Ship
Research & Development Center
Department of the Navy
Bethesda, MD 20084
Attn: T. T. Huang
(Copy No. 7)

Commander
Naval Surface Weapons Center
Department of the Navy
Silver Spring, MD 20910
Attn: Library
(Copy No. 8)

Defense Technical Information
Center
5010 Duke Street
Cameron Station
Alexandria, VA 22314
(Copies 9 through 14)

Naval Research Laboratory
Department of the Navy
Washington, DC 20390
Attn: Library
(Copy No. 15)

Superintendent
Code 1424
Naval Post Graduate School
Monterey, CA 93949
(Copy No. 16)

NASA Lewis Research Center
21000 Brookpark Road
Cleveland, OH 44135
Attn: P. M. Sockol
Code MS 77-5
(Copy No. 17)

Dr. James E. Carter
Computational Fluid Dynamics
United Technologies Research
Center
East Hartford, CT 06108
(Copy No. 18)

DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL MEMORANDUM FILE NO. 84-79, by K. C. Kaufman and G. H. Hoffman, dated 4 May 1984 [continuation]

Dr. Joseph F. Thompson
Department of Aerophysics &
Aerospace Engineering
Mississippi State University
Mississippi State, MS 39762
(Copy No. 19)

Professor C. L. Merkle
Dept. of Mechanical Engineering
The Pennsylvania State University
University Park, PA 16802
(Copy No. 20)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: S. Abdallah
(Copy No. 21)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: J. J. Eisenhuth
(Copy No. 22)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: W. S. Gearhart
(Copy No. 23)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: R. E. Henderson
(Copy No. 24)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: G. H. Hoffman
(Copy No. 25)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: K. C. Kaufman
(Copy No. 26)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: G. C. Lauchle
(Copy No. 27)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: M. W. McBride
(Copy No. 28)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: B. R. Parkin
(Copy No. 29)

DISTRIBUTION LIST FOR UNCLASSIFIED TECHNICAL MEMORANDUM FILE NO. 84-79,
by K. C. Kaufman and G. H. Hoffman, dated 4 May 1984 [continuation]

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: ARL/PSU Library
(Copy No. 30)

Director
Applied Research Laboratory
The Pennsylvania State University
Post Office Box 30
State College, PA 16804
Attn: GTWT Files
(Copy No. 31)

AN INVERSE LAMINAR BOUNDARY LAYER PROBLEM WITH ASSIGNED WALL SHEAR: THE MECHUL FUNCTION REVISITED

K. C. Kaufman and G. H. Hoffman

Technical Memorandum
File No. TM 84-79
4 May 1984
Contract N00024-79-C-6043

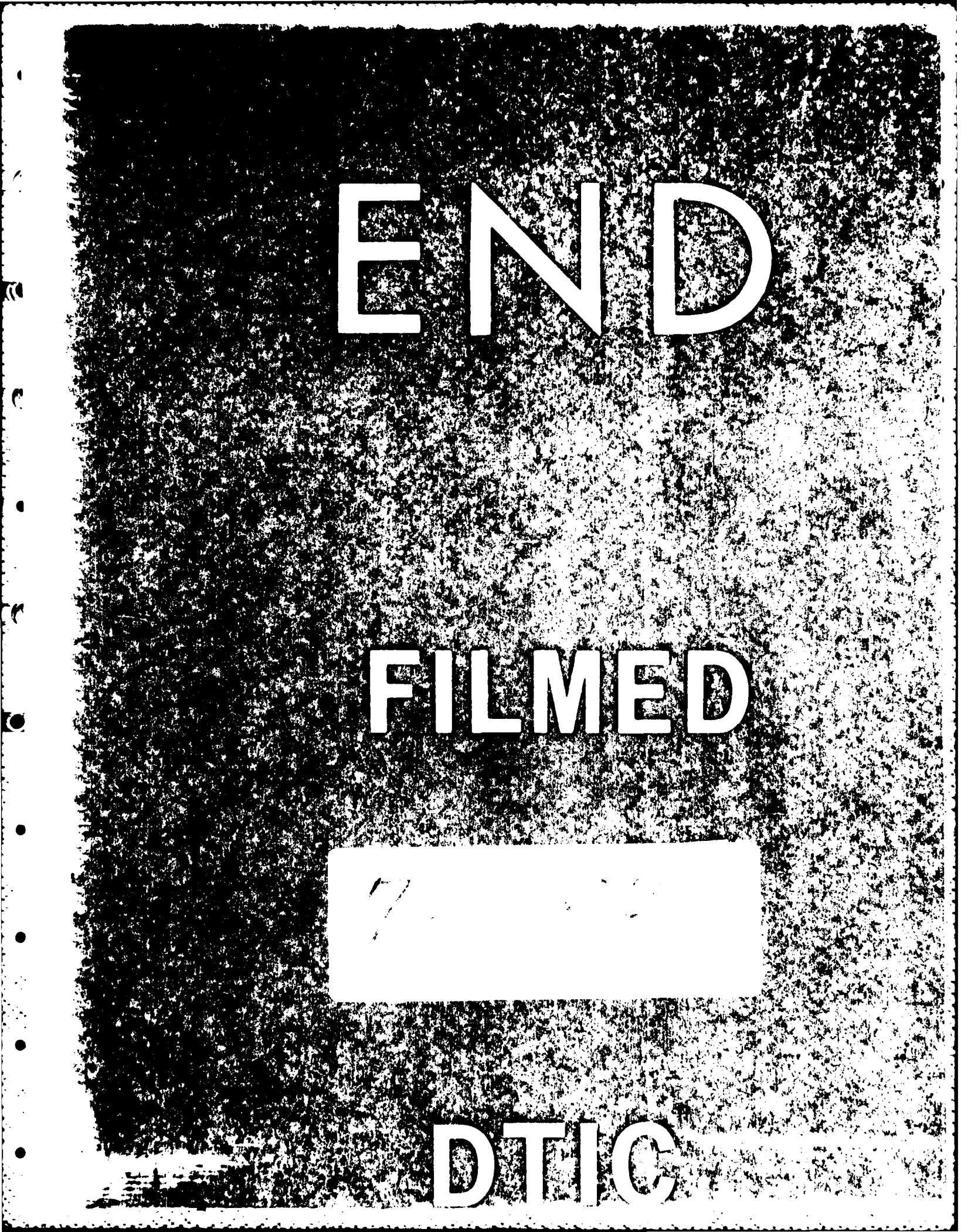
Copy No. ___

The Pennsylvania State University
Intercollege Research Programs and Facilities
APPLIED RESEARCH LABORATORY
Post Office Box 30
State College, PA 16804

Approved for Public Release
Distribution Unlimited

NAVY DEPARTMENT

NAVAL SEA SYSTEMS COMMAND


END
FILMED
DTIC